West Seneca Deer Task Force Agenda

September 21, 2021

The Deer Task Force meeting will be held in the Community Room at the Community Center & Library, 1300 Union Road at 6:00 PM.

- Welcome & New members introduce themselves
- Approval of Meeting Minutes, July 20, 2021 (6:05-6:10)
- Members of the Task Force present on findings from the Town Board's resolutions
  - Jonathan Brotka: WHEREAS, there is a significant lack of community knowledge about the complexity of addressing deer in suburban environments. What information do we want to share with the community about the complexity of these issues? Presentation and discussion (6:10-6:35)
- Review of communications received (emails 6:35-6:40)
- Open for the public to speak (6:40-7:00)
- Adjourn for a work session

Note that the next opportunity for public comment at a Deer Task Force meeting will be Tuesday, November 16.

Resident Concern

Brian Adams <badams@TWSNY.org>
Tue 8/3/2021 8:16 AM
To: Deer Task Force <WSDeerTaskForce@twsny.org>

Dear Members of the Deer Taskforce,

I had received a call the other day from Carol a resident of Summit Ave. regarding issues with deer in the community. Her biggest issue was the deer are eating her shrubs and flowers. She also stated that while driving down Center Road one had almost ran into the side of her car and its becoming a safety issue. I told her I would pass along her information and concerns to the deer taskforce for review. If anyone would like to reach out to her for more information her phone number is 716-984-2945.

Sincerely,

Brian J. Adams

Superintendent of Highways
Town of West Seneca

39 South Ave.
West Seneca, NY 14224
Phone: 716-674-4850

Re: Deer feeding from residence

Marlene Yahoo <lladychemist@yahoo.com>
Mon 8/2/2021 1:21 PM
To: Deer Task Force <WSDeerTaskForce@twsny.org>

Hi Cynnie:

Yes I am asking how to get residents to not feed deer which in itself, if it had never started, would have deterred them from gathering in yards looking for food. On Norwood there have been many deer crossing the street and there was almost a collision Saturday night here right near my home. An elderly couple stopped in time too. Was quite upsetting to me considering the size of the doe that came across. My husband and I have encountered them in the early hours of the workday just standing in the street too. There are now 2-3 bucks, 6-7 doe and 5-6 fawn/yearlings.

I realize from the first task force meeting I was able to attend it was mentioned through the Cornell person that it is a 5-6 year task. Right now Norwood has an issue with deer, specifically at the Union road end near Alton. Not sure if this was ever brought to the task force but it is an issue. Can a generic letter go out to residence of Norwood ?

Thanks for your help. Feel free to contact me at 983-5311.

Marlene Davis

On Aug 2, 2021, at 10:37 AM, Deer Task Force <WSDeerTaskForce@twsny.org> wrote:

Hello Marlene, Thank you for your email.

I am not sure what you are asking. Are you asking how to deter deer? Or are you asking about how to ask someone to STOP feeding deer?

Cynnie
Facilitator,
West Seneca Deer Task Force

From: Marlene Yahoo <lladychemist@yahoo.com>
Sent: Monday, August 2, 2021 7:04 AM
To: Deer Task Force <WSDeerTaskForce@twsny.org>
Subject: Deer feeding from residence

GoodSay:

Can you inform me as to whether there is any action that can be taken for feeding deer from a residence?

Thank you

Marlene Davis

Re: Deer feeding from residence

Deer Task Force <WSDeerTaskForce@twsny.org>
Mon 8/2/2021 4:53 PM
To: Marlene Yahoo <lladychemist@yahoo.com>

Thank you for asking about this. We are discussing an education program around these very issues.

As you know, the Task Force will take some time, but I will forward this to Gary Dickson.

Cynnie

From: Marlene Yahoo <lladychemist@yahoo.com>
Sent: Monday, August 2, 2021 1:21 PM
To: Deer Task Force <WSDeerTaskForce@twsny.org>
Subject: Re: Deer feeding from residence

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Biology of food, water, predator / prey relations, human development and encroachment on wetlands, breeding rates, family size.

James Burnette <jhburnette@hotmail.com>
Mon 7/19/2021 8:03 PM
To: Deer Task Force <WSDeerTaskForce@twsny.org>

7 attachments (6 MB)
Competing Species -- Non-linear DE's pg. 1 of 7.jpg; Competing Species -- Non-linear DE's pg. 2 of 7.jpg; Competing Species -- Non-linear DE's pg. 3 of 7.jpg; Competing Species -- Non-linear DE's pg. 4 of 7.jpg; Competing Species -- Non-linear DE's pg. 5 of 7.jpg; Competing Species -- Non-linear DE's pg. 6 of 7.jpg; Competing Species -- Non-linear DE's pg. 7 of 7.jpg;

West Seneca Deer Task Force

bcc: Professor Robin Foster, Canisius College

bcc: Saundra Mercado

Mon, 19 July, 2021

Hello:

I would like to submit this e-mail for tomorrow's public meeting with the West Seneca Deer Task Force.

The initial 18 May meeting went smoothly, except for a snarky comment muttered by a fellow in the audience. Apparently, he didn't agree with the findings that an abrupt reduction in deer population leads to larger subsequent number of offspring; a statement made by Saundra Mercado. (Deer contraception and family planning will be brought up **by** someone else, or **at** a later meeting. I suspect much of the concern here is with lethal vs. non-lethal population control methods.)

My concern today is the quality of arguments and references, and with the decorum of the audience. *"Bull sh*t"* is not a proper rebuttal. It may be the opening line of one, but it does not constitute a good argument. I therefore wish to attach pages (best viewed with Windows Picture & FAX Viewer ?) from Boyce & DiPrima, *Elementary Differential Equations & Boundry Value Problems, 4th ed,* Wiley, 1986. Though the emphasis is on math for scientists and engineers, the explanations in the Predator - Prey relations section starting on p.493 are still relevant, and the references cited by the authors may be of value, as they are not limited to analysis by pointy headed academics in ivory towers, but include data from the Hudson Bay Company of Canada (these guys were purely profit oriented, and no one can accuse their data of having a tree hugger bias).

Though the math may not be of interest to anyone on the Task Force other than the Professor, it serves a purpose once enough data is collected. (I'm certain other audience members will speak on the population data collected from neighboring towns.) Even if you skip over the math, the salient features of the biology are relevant to our discussions today, qualitatively, if not quantitatively. That is, we'll get the general idea of population dynamics, and what happens when excess pressure from the rural ag community demands that the NYS DEC allow open season on our friends, the coyote and wild cats (possibly the only real solutions to the "deer problem").

The relation to food and brood size is not mentioned in the text. This may require other census sources as well as agricultural and NOAA weather data for specific years. I will start asking around, beginning with NYS DEC.

Yours truly,

James H. Burnette
64 St.Johns Place
Lackawanna, NY 14218

P.S. How did the task force verify the actual deer population and extent of garden / agricultural damage and automobile collisions, as well as health / disease state of killed deer ? We cannot rely on hearsay, and the method of data collection must also be discussed for the public to understand -- otherwise, we won't know if we're making any progress with simple fixes such as levying fines against residents who feed wildlife, either intentionally or by careless overflow of garbage totes. This would be a very easy first step, and I look forward to the metrics in a few months.

JHB

19. Consider the equations for two competing species derived in Section 9.1:

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y), \qquad dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x).$$

Suppose that $\epsilon_1/\sigma_1 < \epsilon_2/\alpha_2$ and $\epsilon_2/\sigma_2 < \epsilon_1/\alpha_1$.
(a) Find the critical point (X, Y) for which both species can coexist.
(b) By making the change of variables $x = X + u$, $y = Y + v$ transform the system of equations to one with a critical point at $u = 0$, $v = 0$. Observe that the system is almost linear.
(c) Classify the critical point as to type and stability.

20. Carry out the calculations of Problem 19 for the case $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$.

9.4 Competing Species and Predator–Prey Problems

In this section we consider two problems in ecology: competing species and predator–prey.

COMPETING SPECIES. In Section 9.1 we showed that a model for the competition between two species with population densities x and y leads to the differential equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y),$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x), \qquad (1)$$

where the parameters $\epsilon_1, \sigma_1, \ldots, \alpha_2$ are positive. As we saw then, we can analyze these equations by dividing the phase plane into regions according to the signs of dx/dt and dy/dt and then drawing typical trajectories. Let us now see how we can use the theory of almost linear systems to obtain a more precise understanding of what happens.

We start by considering the following specific example:

$$dx/dt = x(1 - x - y),$$
$$dy/dt = y(0.5 - 0.75x - 0.25y), \qquad (2)$$

or

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x^2 + xy \\ 0.75xy + 0.25y^2 \end{pmatrix}. \qquad (3)$$

For convenience we think of x and y as the population densities of two species of bacteria competing with each other for the same supply of food. We ask whether there are equilibrium states that might be reached, or whether a periodic growth and decay will be observed, and how such possibilities depend on the initial state of the two cultures.

The critical points of the system (2) are the solutions of the nonlinear algebraic equations

$$x(1 - x - y) = 0,$$
$$y(0.5 - 0.75x - 0.25y) = 0. \qquad (4)$$

There are four critical points, namely, (0,0), (1,0), (0,2), and (0.5,0.5). The system (2) is almost linear in the neighborhood of each critical point, so we investigate the trajectories near each of these points by considering the corresponding linear system.

$x = 0$, $y = 0$. This corresponds to a state in which both bacteria die as a result of their competition. From Eq. (3) the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad (5)$$

whose eigenvalues and eigenvectors are

$$r_1 = 1, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = 0.5, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad (6)$$

Thus the general solution of the system (5) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{t/2}. \qquad (7)$$

The origin is an unstable improper node of both the linear system (5) and the nonlinear system (3). In the neighborhood of the origin all the trajectories are tangent to the y axis except for one pair of trajectories that lies along the x axis. Since the origin is an unstable critical point, this equilibrium solution will not occur in practice.

$x = 1$, $y = 0$. This corresponds to a state in which bacteria x survives the competition but bacteria y does not. To examine this critical point let $x = 1 + u$, $y = 0 + v$. Substituting for x and y in Eqs. (2) or (3) and simplifying, we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^2 + uv \\ 0.75uv + 0.25v^2 \end{pmatrix}. \qquad (8)$$

The corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \qquad (9)$$

Its eigenvalues and eigenvectors are

$$r_1 = -1, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad r_2 = -0.25, \quad \xi^{(2)} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}. \qquad (10)$$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 4 \\ -3 \end{pmatrix}e^{-t/4}. \tag{11}$$

The point (1, 0) is an asymptotically stable improper node of the linear system (9) and hence of the nonlinear systems (8) and (3) as well. If the initial values of x and y are sufficiently close to (1, 0), then the interaction process will lead ultimately to that state.

The behavior of the trajectories near the point (1, 0) can be seen from Eq. (11). If $c_2 = 0$, then there is one pair of trajectories that approaches the critical point along the x axis. All other trajectories approach (1, 0) tangent to the line (with slope $-\frac{3}{4}$) that is determined by the eigenvector $\xi^{(2)}$.

$\boldsymbol{x = 0,\ y = 2.}$ In this case bacteria y survives but bacteria x does not. The analysis is similar to that for the point (1, 0). If we let $x = u$, $y = 2 + v$, then we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1.5 & -0.5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^2 + uv \\ 0.75uv + 0.25v^2 \end{pmatrix}. \tag{12}$$

The eigenvalues and eigenvectors of the corresponding linear system are

$$r_1 = -1, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \qquad r_2 = -0.5, \quad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{13}$$

and its general solution is

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-t/2}. \tag{14}$$

The critical point (0, 2) is an asymptotically stable improper node. All trajectories approach the critical point along the y axis except for one pair that approaches along the line with slope 3.

$\boldsymbol{x = 0.5,\ y = 0.5.}$ This critical point corresponds to a mixed equilibrium state or coexistence; a standoff, so to speak, in the competition between the two bacteria cultures. To examine the nature of this critical point we let $x = 0.5 + u$, $y = 0.5 + v$. Substituting for x and y in Eqs. (2) or (3), we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.375 & -0.125 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^2 + uv \\ 0.75uv + 0.25v^2 \end{pmatrix}. \tag{15}$$

The eigenvalues and eigenvectors of the corresponding linear system are

$$r_1, r_2 = (-5 \pm \sqrt{57})/16; \qquad \xi^{(1)}, \xi^{(2)} = \begin{pmatrix} 1 \\ (-3 \pm \sqrt{57})/8 \end{pmatrix}. \tag{16}$$

Since the eigenvalues are of opposite sign, the critical point (0.5, 0.5) is a saddle point, and hence is unstable. One pair of trajectories approaches the critical point as $t \to \infty$; the others recede from it. The entering trajectories have the slope $(\sqrt{57} - 3)/8 \cong 0.57$ determined from the eigenvector associated with the negative eigenvalue.

A diagram of what the trajectories look like in the neighborhood of each critical point is shown in Figure 9.25*a*. With a little additional work it is possible to extend the local pictures and obtain a global picture of the trajectories in the phase plane. First, we are only interested in x and y positive. Since trajectories cannot cross other trajectories, and since the x and y axes are trajectories, it follows that a trajectory that starts in the first quadrant must stay in the first quadrant, and a trajectory that starts in any other quadrant cannot enter the first quadrant. Second, we accept without proof two facts that follow from more advanced theory: (i) the system (2) does not have any periodic solutions, that is, trajectories that are closed curves; and (ii) a trajectory that is not a closed curve must either enter a critical point or recede to infinity as $t \to \infty$. For x and y large the nonlinear terms $-(x^2 + xy)$ and $-\frac{1}{4}(y^2 + 3xy)$ in the first and second of Eqs. (2), respectively, outweigh the linear terms. Since they are negative, dx/dt and dy/dt are negative for x and y large. Thus for x and y large the direction of motion on every trajectory is inward. The trajectories cannot escape to infinity! Eventually they must head toward one or the other of the two stable nodes. The *schematic* sketch shown in Figure 9.25*b* is not an unreasonable representation[4] of what must be happening in the first quadrant. If the initial values of x and y are in region I of Figure 9.25*b*, then x wins the competition; if the initial values are in region II, then y wins. "Peaceful coexistence" is not possible unless the initial point lies exactly on the dividing trajectory (separatrix). Of particular interest would be the determination of the dividing trajectories that enter the saddle point (0.5, 0.5) which separate regions I and II.

Now let us return to the general system (1). Recall from Section 9.1 that four cases must be considered depending on the relative orientation of the lines

$$\epsilon_1 - \sigma_1 x - \alpha_1 y = 0 \quad \text{and} \quad \epsilon_2 - \sigma_2 y - \alpha_2 x = 0, \tag{17}$$

as shown in Figure 9.26. Let (X, Y) denote any critical point in any one of the four cases. To study the system (1) in the neighborhood of this critical point we let

$$x = X + u, \qquad y = Y + v, \tag{18}$$

[4] The dotted trajectories that appear to go, for example, from (0, 0) to (1, 0) actually correspond to trajectories through particular initial points. As $t \to \infty$ they approach the stable node (1, 0); as $t \to -\infty$ they approach the unstable node (0, 0).

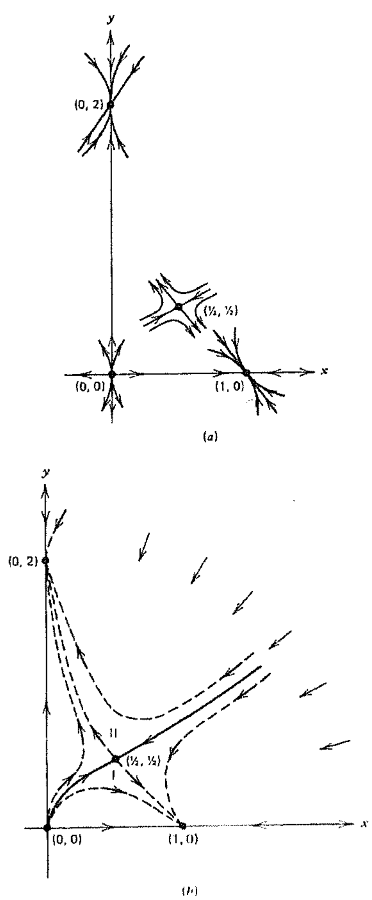

FIGURE 9.25 (a) Trajectories in the neighborhood of each critical point of the system (2). (b) Overall pattern of trajectories for the system (2).

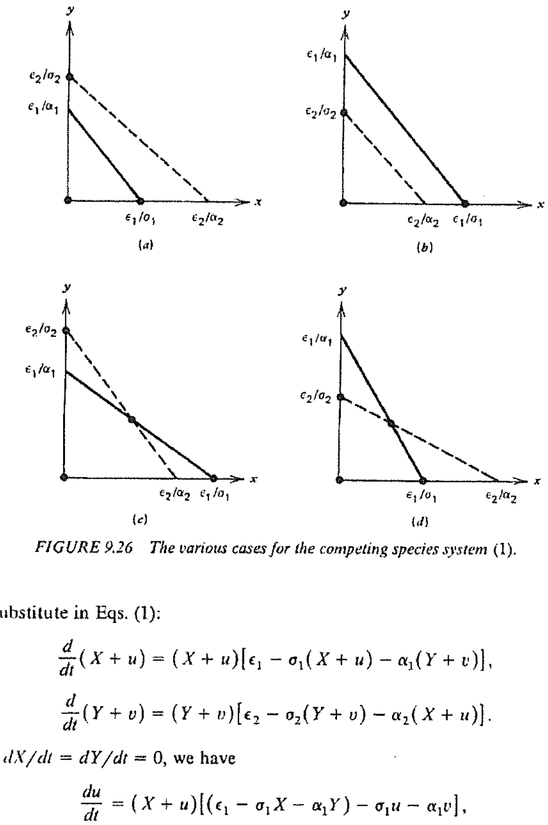

FIGURE 9.26 The various cases for the competing species system (1).

and substitute in Eqs. (1):

$$\frac{d}{dt}(X+u) = (X+u)\left[\epsilon_1 - \sigma_1(X+u) - \alpha_1(Y+v)\right],$$
$$\frac{d}{dt}(Y+v) = (Y+v)\left[\epsilon_2 - \sigma_2(Y+v) - \alpha_2(X+u)\right]. \tag{19}$$

Since $dX/dt = dY/dt = 0$, we have

$$\frac{du}{dt} = (X+u)\left[(\epsilon_1 - \sigma_1 X - \alpha_1 Y) - \sigma_1 u - \alpha_1 v\right], \tag{20a}$$

$$\frac{dv}{dt} = (Y+v)\left[(\epsilon_2 - \sigma_2 Y - \alpha_2 X) - \sigma_2 v - \alpha_2 u\right]. \tag{20b}$$

The right side of Eq. (20a) has the form $X(\epsilon_1 - \sigma_1 X - \alpha_1 Y) + (\ )u + (\ )v + (\ )u^2 + (\ )uv$. The constant term is zero since either $X = 0$ or $\epsilon_1 - \sigma_1 X - \alpha_1 Y = 0$. Similarly, the right side of Eq. (20b) reduces to the form $(\ )u + (\ )v + (\ )uv + (\ )v^2$. Next, it is clear that Eqs. (20) are almost linear, so we consider the corresponding linear system

$$\begin{aligned} du/dt &= \left[(\epsilon_1 - \sigma_1 X - \alpha_1 Y) - \sigma_1 X\right] u - \alpha_1 X v, \\ dv/dt &= -\alpha_2 Y u + \left[(\epsilon_2 - \sigma_2 Y - \alpha_2 X) - \sigma_2 Y\right] v. \end{aligned} \tag{21}$$

Equations (21), along with Theorem 9.2 of Section 9.3, can be used to determine the type and stability of any critical point (X, Y) of the original system (1). These linear equations are referred to as the linearized equations for small perturbations in the neighborhood of the critical point (X, Y). The process of deriving them is called linearization.

We use Eqs. (21) to determine whether the model given by the system (1) can ever lead to coexistence for the two species x and y, and if so, under what conditions on the parameters $\epsilon_1, \sigma_1, \ldots, \alpha_2$. The four possible situations are shown in Figure 9.26; coexistence is possible only in cases (c) and (d). The values of $X \neq 0$ and $Y \neq 0$ are obtained by solving the system of simultaneous linear algebraic equations (17). We readily obtain

$$X = \frac{\epsilon_1 \sigma_2 - \epsilon_2 \alpha_1}{\sigma_1 \sigma_2 - \alpha_1 \alpha_2}, \qquad Y = \frac{\epsilon_2 \sigma_1 - \epsilon_1 \alpha_2}{\sigma_1 \sigma_2 - \alpha_1 \alpha_2}. \tag{22}$$

Moreover, since $\epsilon_1 - \sigma_1 X - \alpha_1 Y = 0$ and $\epsilon_2 - \sigma_2 Y - \alpha_2 X = 0$, Eqs. (21) immediately reduce to

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\sigma_1 X & -\alpha_1 X \\ -\alpha_2 Y & -\sigma_2 Y \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{23}$$

The eigenvalues of the system (23) are found from the equation

$$r^2 + (\sigma_1 X + \sigma_2 Y) r + (\sigma_1 \sigma_2 XY - \alpha_1 \alpha_2 XY) = 0. \tag{24}$$

Thus

$$r = \frac{-(\sigma_1 X + \sigma_2 Y) \pm \sqrt{(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1 \sigma_2 - \alpha_1 \alpha_2) XY}}{2}. \tag{25}$$

If $\sigma_1 \sigma_2 - \alpha_1 \alpha_2 < 0$ then the radicand of Eq. (25) is positive and greater than $(\sigma_1 X + \sigma_2 Y)^2$. Thus the eigenvalues are real and of opposite sign. Consequently, the critical point (X, Y) is an unstable saddle point, and coexistence is not possible. This is case in the specific example given by Eqs. (2); there $\sigma_1 = 1$, $\alpha_1 = 1$, $\sigma_2 = \frac{1}{4}$, $\alpha_2 = \frac{3}{4}$ and $\sigma_1 \sigma_2 - \alpha_1 \alpha_2 = -\frac{1}{2}$.

On the other hand, if $\sigma_1 \sigma_2 - \alpha_1 \alpha_2 > 0$, then the radicand of Eq. (25) is less than $(\sigma_1 X + \sigma_2 Y)^2$. Thus the eigenvalues are real, negative, and unequal, or complex with negative real part. A simple analysis of the radicand of Eq. (25) shows that the eigenvalues cannot be complex (see Problem 6). Thus the critical point is an asymptotically stable improper node. Coexistence is possible if $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$.

Let us relate this result to Figures 9.26c and 9.26d. In Figure 9.26c we have

$$\frac{\epsilon_1}{\sigma_1} > \frac{\epsilon_2}{\alpha_2} \quad \text{or} \quad \epsilon_1 \alpha_2 > \epsilon_2 \sigma_1 \quad \text{and} \quad \frac{\epsilon_2}{\sigma_2} > \frac{\epsilon_1}{\alpha_1} \quad \text{or} \quad \epsilon_2 \alpha_1 > \epsilon_1 \sigma_2. \tag{26}$$

These inequalities coupled with the condition that X and Y given by Eqs. (22) be positive yield the inequality $\sigma_1 \sigma_2 < \alpha_1 \alpha_2$. Hence in this case the mixed state is an unstable saddle point. Corresponding to Figure 9.26d, we have

$$\frac{\epsilon_2}{\alpha_2} > \frac{\epsilon_1}{\sigma_1} \quad \text{or} \quad \epsilon_2 \sigma_1 > \epsilon_1 \alpha_2 \quad \text{and} \quad \frac{\epsilon_1}{\alpha_1} > \frac{\epsilon_2}{\sigma_2} \quad \text{or} \quad \epsilon_1 \sigma_2 > \epsilon_2 \alpha_1. \tag{27}$$

Now, the condition X and Y positive yields $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$. Hence this mixed state is asymptotically stable. For this case we can also show that the critical points $(0, 0)$, $(\epsilon_1/\sigma_1, 0)$ and $(0, \epsilon_2/\sigma_2)$ are unstable. Thus no matter what the initial values of $x \neq 0$ and $y \neq 0$ are, the two species approach an equilibrium state of coexistence given by Eqs. (22).

Equations (1) provide the biological interpretation of the result that $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$ leads to coexistence and $\alpha_1 \alpha_2 > \sigma_1 \sigma_2$ does not allow coexistence. The σ's are a measure of the inhibitory effect the growth of each species has on its own growth rate, while the α's are a measure of the inhibiting effect the growth of each species has on the other species (interaction). Thus when $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$ interaction is "small" and the species can coexist; when $\alpha_1 \alpha_2 > \sigma_1 \sigma_2$ interaction (competition) is "large" and the species cannot coexist—one must die out.

PREDATOR–PREY. As a second example we consider the classical predator–prey problem. We study an ecological situation involving two species, one of which preys on the other (does not compete with it for food but actually preys on it) while the other lives on a different source of food. An example is foxes and rabbits in a closed forest; the foxes prey on the rabbits, the rabbits live on the vegetation in the forest. Other examples are bass in a lake as predators and redear (sunfish) as prey, and lady bugs as predators and aphids (insects that suck the juice of plants) as prey. Let $H(t)$ and $P(t)$ be the populations of prey and predator, respectively, at time t.

We build as simple a model of the interaction as possible. We make the following assumptions:

1. In the absence of the predator the prey grows without bound; thus $dH/dt = aH$, $a > 0$, for $P = 0$.
2. In the absence of the prey the predator dies out, thus $dP/dt = -cP$, $c > 0$, for $H = 0$.
3. The increase in the number of predators is wholly dependent on the food supply (the prey) and the prey are consumed at a rate proportional to the

number of encounters between predators and prey. Thus, for example, if the number of prey is doubled the number of encounters is doubled. Encounters decrease the number of prey and increase the number of predators. A fixed proportion of prey is killed in each encounter, and the rate of population growth of the predator is enhanced by a factor proportional to the amount of prey consumed.

As a consequence, we have the equations

$$\begin{aligned} dH/dt &= aH - \alpha HP = H(a - \alpha P), \\ dP/dt &= -cP + \gamma HP = P(-c + \gamma H). \end{aligned} \tag{28}$$

The constants a, c, α, and γ are positive; a and c are the growth rate of the prey and the death rate of the predator, respectively, and α and γ are measures of the effect of the interaction between the two species. Equations (28) are known as the Lotka-Volterra equations. They were developed in papers by Lotka[5] in 1925 and Volterra[6] in 1926. Although these equations are simple, they do characterize a wide class of problems. Ways of making them more realistic are discussed at the end of this section and in the problems.

What happens for given initial values of $P > 0$ and $H > 0$? Will the predators eat all of their prey and in turn die out, will the predators die out because of a too low level of prey and then the prey grow without bound, will an equilibrium state be reached, or will a cyclic fluctuation of prey and predator occur?

The critical points of Eqs. (28) are the solutions of

$$H(a - \alpha P) = 0, \qquad P(-c + \gamma H) = 0. \tag{29}$$

These solutions are $H = 0$, $P = 0$ and $H = c/\gamma$, $P = a/\alpha$. It is easy to show that the critical point $(0, 0)$ is a saddle point, and hence unstable. Entrance to the saddle point is along the line $H = 0$; all other trajectories recede from the critical point.

[5]Alfred J. Lotka (1880–1949), an American biophysicist, was born in what is now the Ukraine, and was educated mainly in Europe. He is remembered chiefly for his formulation of the Lotka-Volterra equations. He was also the author, in 1924, of the first book on mathematical biology; it is now available as *Elements of Mathematical Biology* (New York: Dover, 1956).

[6]Vito Volterra (1860–1940), a distinguished Italian mathematician, held professorships at Pisa, Turin, and Rome. He is particularly famous for his work in integral equations and functional analysis. Indeed, one of the major classes of integral equations is named for him; see Problem 14 of Section [illegible]. His theory of interacting species was motivated by data collected by a friend, D'Ancona, concerning fish catches in the Adriatic Sea. A translation of his 1926 paper can be found in an appendix of R. N. Chapman, *Animal Ecology with Special Reference to Insects* (New York: McGraw-Hill, 1931).

To study the critical point $(c/\gamma, a/\alpha)$ we let

$$H = (c/\gamma) + u, \qquad P = (a/\alpha) + v. \tag{30}$$

Substituting for H and P in Eqs. (28), we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -\alpha uv \\ \gamma uv \end{pmatrix}. \tag{31}$$

The system (31) is almost linear, and the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{32}$$

The eigenvalues of the system (32) are $r = \pm i\sqrt{ac}$, so the critical point is a (stable) center of the linear system. To find the trajectories of the system (32) we can divide the second equation by the first:

$$\frac{dv}{du} = \frac{dv/dt}{du/dt} = -\frac{(\gamma a/\alpha)u}{(\alpha c/\gamma)v}, \tag{33}$$

or

$$(\gamma a/\alpha)u\,du + (\alpha c/\gamma)v\,dv = 0. \tag{34}$$

Consequently,

$$(\gamma a/\alpha)u^2 + (\alpha c/\gamma)v^2 = k, \tag{35}$$

where k is an arbitrary nonnegative constant of integration. Thus the trajectories are ellipses, a few of which are sketched in Figure 9.27.

While the critical point is a stable center of the linear system (32), we need to assess its character for the almost linear system (31). Here, as we know, our theory for almost linear systems fails. The effect of the nonlinear terms may be to

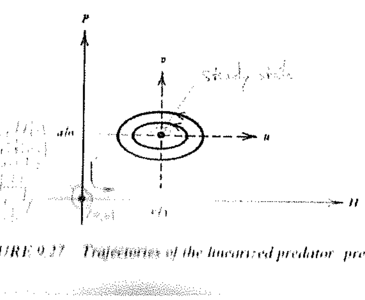

FIGURE 9.27 Trajectories of the linearized predator–prey equation.

change the center into a stable spiral point, or into an unstable spiral point, or it may remain as a stable center. Fortunately, for the predator–prey problem we can actually solve the nonlinear equations (28) and determine what happens. Dividing the second of Eqs. (28) by the first equation, we obtain

$$\frac{dP}{dH} = \frac{P(-c + \gamma H)}{H(a - \alpha P)}. \tag{36}$$

Upon separating the variables in Eq. (36), we have

$$\frac{a - \alpha P}{P}\, dP = \frac{-c + \gamma H}{H}\, dH,$$

from which it follows that

$$a \ln P - \alpha P = -c \ln H + \gamma H + \ln C, \tag{37}$$

where C is a constant of integration. We cannot solve Eq. (37) explicitly for P in terms of H or for H in terms of P, but it can be shown that the graph of this equation for a fixed value of C is a closed curve (not an ellipse, of course) enclosing the critical point $(c/\gamma, a/\alpha)$.[7] Thus the predator and prey have a cyclic variation about the critical point and the critical point is also a center of the nonlinear system.

We can analyze this cyclic variation in more detail when the deviation from the point $(c/\gamma, a/\alpha)$ is small; that is, when it is permissible to linearize the perturbation equations (31) for u and v. As we have noted, the trajectories are the family of ellipses given by Eq. (35). We can also verify, either by solving Eqs. (32) or by direct substitution, that the solution of Eqs. (32) is

$$u(t) = \frac{c}{\gamma} K \cos(\sqrt{ac}\, t + \phi), \qquad v(t) = \frac{a}{\alpha}\sqrt{\frac{c}{a}}\, K \sin(\sqrt{ac}\, t + \phi), \tag{38}$$

where the constants K and ϕ are determined by the initial conditions. Thus

$$\begin{aligned} H(t) &= \frac{c}{\gamma} + \frac{c}{\gamma} K \cos(\sqrt{ac}\, t + \phi), \\ P(t) &= \frac{a}{\alpha} + \frac{a}{\alpha}\sqrt{\frac{c}{a}}\, K \sin(\sqrt{ac}\, t + \phi). \end{aligned} \tag{39}$$

These equations are valid for the elliptical trajectories close to the critical point $(c/\gamma, a/\alpha)$. We can use them to draw several conclusions about the cyclic variation of the predator and prey on such trajectories.

[7] Volterra gave a clever elementary geometric proof of this result. Shorter (but mathematically more advanced) proofs have also been discovered.

1. The size of the predator and prey populations varies sinusoidally with period $2\pi/\sqrt{ac}$. This period of oscillation is independent of the initial conditions.
2. The predator and prey populations are out of phase by one-quarter of a cycle. The prey leads and the predator lags as one might expect. This is discussed in Problem 9.
3. The amplitudes of the oscillations are Kc/γ for the prey and $a\sqrt{c}\,K/\alpha\sqrt{a}$ for the predator and hence depend on the initial conditions as well as the parameters of the problem.
4. The average numbers of predators and prey over one complete cycle are c/γ and a/α, respectively. These are the same as the equilibrium populations (see Problem 10).

Cyclic variations of predator and prey as predicted by Eqs. (28) have been observed in nature. One striking example is described by Odum (pp. 191–192): based on the records of the Hudson Bay Company of Canada, the abundance of lynx and snowshoe hare as indicated by the number of pelts turned in over the period 1845–1935 shows a distinct periodic variation with a period of nine to ten years. The peaks of abundance are followed by very rapid declines, and the peaks of abundance of the lynx and hare are out of phase, with that of the hare preceding that of the lynx by a year or more.

The Volterra-Lotka model of the predator–prey problem has revealed a cyclic variation that was perhaps intuitively expected. On the other hand, the use of the Volterra-Lotka model in other situations can lead to conclusions that are not intuitively obvious. An example revealing a possible danger in using insecticides is given in Problem 12.

One criticism of the Volterra-Lotka predator–prey model is that in the absence of the predator the prey will grow without bound. This can be corrected by allowing for the natural inhibiting effect that an increasing population has on the growth rate of the population; for example, by modifying the first of Eqs. (28) so that when $P = 0$ it reduces to a logistic equation for H (see Problem 13). The models of two competing species and predator–prey discussed here can be modified to allow for the effect of time delays; probabilistic and statistical effects can also be included. Finally, we mention that there are *discrete* analogs of each of the problems we have discussed corresponding to species that breed only at certain times. The mathematics of the discrete problems are often interesting and some of the results are unexpected. These generalizations are discussed in the references given at the end of this chapter as well as in other books on mathematical biology and ecology.

We conclude with a final warning. Using only elementary phase plane theory for one and two nonlinear ordinary differential equations, we have been able to illustrate several of the fundamental principles of simple biological systems. But one should not be misled—ecology is not this simple.

PROBLEMS

Each of Problems 1 through 5 can be interpreted as describing the interaction of two species with population densities x and y. In each of these problems
(a) Find the critical points.
(b) For each critical point find the corresponding linear system. Find the eigenvalues of the linear system, and determine the type and stability of each critical point.
(c) Sketch the trajectories in the neighborhood of each critical point. Determine the limiting behavior of x and y as $t \to \infty$.

1. $dx/dt = x(1 - x + 0.5y)$
   $dy/dt = y(2.5 - 1.5y + 0.25x)$
2. $dx/dt = x(1.5 - x - 0.5y)$
   $dy/dt = y(2 - y - 0.75x)$
3. $dx/dt = x(1 - 0.5x - 0.5y)$
   $dy/dt = y(-0.25 + 0.5x)$
4. $dx/dt = x(1.5 - x - 0.5y)$
   $dy/dt = y(2 - 0.5y - 1.5x)$
5. $dx/dt = x(1.125 - x - 0.5y)$
   $dy/dt = y(-1 + x)$
6. Show that

$$(\sigma_1 X + \sigma_2 Y)^2 - 4(\sigma_1\sigma_2 - \alpha_1\alpha_2)XY = (\sigma_1 X - \sigma_2 Y)^2 + 4\alpha_1\alpha_2 XY.$$

   Hence conclude that the eigenvalues given by Eq. (25) can never be complex.
7. Two species of fish that compete with each other for food, but do not prey on each other, are bluegill and redear. Suppose that a pond is stocked with bluegill and redear, and let x and y be the populations of bluegill and redear, respectively, at time t. Suppose further that the competition is modeled by the equations

$$dx/dt = x(\epsilon_1 - \sigma_1 x - \alpha_1 y),$$
$$dy/dt = y(\epsilon_2 - \sigma_2 y - \alpha_2 x).$$

   (a) If $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_2/\sigma_2 > \epsilon_1/\alpha_1$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen?
   (b) If $\epsilon_1/\sigma_1 > \epsilon_2/\alpha_2$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, show that the only equilibrium populations in the pond are no fish, no redear, or no bluegill. What will happen?
8. Consider the competition between bluegill and redear mentioned in Problem 7. Suppose that $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, so, as shown in the text, there is a stable equilibrium point at which both species can coexist. It is convenient to rewrite the equations of Problem 7 in terms of the carrying capacities of the pond for bluegill ($B = \epsilon_1/\sigma_1$) in the absence of redear and for redear ($R = \epsilon_2/\sigma_2$) in the absence of bluegill.
   (a) Show that the equations of Problem 7 take the form

$$\frac{dx}{dt} = \epsilon_1 x\left(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y\right), \qquad \frac{dy}{dt} = \epsilon_2 y\left(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x\right),$$

   where $\gamma_1 = \alpha_1/\sigma_1$ and $\gamma_2 = \alpha_2/\sigma_2$. Determine the coexistence equilibrium point (X, Y) in terms of B, R, γ_1, and γ_2.

   (b) Now suppose that a fisherman fishes only for bluegill with the effect that B is reduced. What effect does this have on the equilibrium populations? Is it possible, by fishing, to reduce the population of bluegill to such a level that they will die out?
9. In this problem we examine the phase difference between the cyclic variations of the predator and prey populations as given by Eqs. (39) of the text. Suppose we assume that $K > 0$ and that T is measured from the time that the prey population (H) is a maximum; then $\phi = 0$. Show that the predator population (P) is a maximum at $t = \pi/2\sqrt{ac} = T/4$, where T is the period of the oscillation. When is the prey population increasing most rapidly, decreasing most rapidly, a minimum? Answer the same questions for the predator population. Draw a typical elliptic trajectory enclosing the point $(c/\gamma, a/\alpha)$, and mark these points on it.
10. The average sizes of the prey and predator populations are defined as

$$\overline{H} = \frac{1}{T}\int_A^{A+T} H(t)\,dt, \qquad \overline{P} = \frac{1}{T}\int_A^{A+T} P(t)\,dt,$$

   respectively, where T is the period of a full cycle and A is any nonnegative constant. Show for trajectories near the critical point that $\overline{H} = c/\gamma$, $\overline{P} = a/\alpha$.
11. Suppose that the predator–prey equations (28) of the text govern foxes (P) and rabbits (H) in a forest. A trapping company is engaging in trapping foxes and rabbits for their pelts. Explain why it is reasonable for the company to conduct its operation in such a way as to move the population of each species closer to the center $(c/\gamma, a/\alpha)$. When is it best to trap foxes? Rabbits? Rabbits and foxes? Neither? *Hint:* See Problem 9. A mathematical argument is not required.
12. Suppose that an insect population (H) is controlled by a natural predator population (P) according to the model (28), so that there are small cyclic variations of the populations about the critical point $(c/\gamma, a/\alpha)$. Show that it is self-defeating to employ an insecticide if the insecticide also kills the predator. Assume that the insecticide kills both prey and predator at rates proportional to each population, respectively. To ban insecticides on the basis of this very simple model would certainly be ill-advised. On the other hand, it is also rash to ignore the possible genuine existence of a phenomenon suggested by a simple model.
13. As was mentioned in the text, one improvement in the predator–prey model is to modify the equation for the prey so that it has the form of a logistic equation in the absence of the predator. Thus in place of Eqs. (28) we consider the model system

$$dH/dt = H(a - \sigma H - \alpha P),$$
$$dP/dt = P(-c + \gamma H),$$

   where a, σ, α, c, and γ are positive constants. Determine all critical points and discuss their nature and stability. Assume that $a/\sigma \gg c/\gamma$. What happens for initial data $H \neq 0$, $P \neq 0$?

9.5 Liapunov's Second Method

In Section 9.3 we showed how the stability of a critical point of an almost linear system can usually be determined from a study of the corresponding linear system. However, no conclusion can be drawn when the critical point is a center

Deer are overtaking my backyard. HELP!

MD Jones <DMJones_5@hotmail.com>
Mon 8/16/2021 1:21 PM
To: Deer Task Force <WSDeerTaskForce@twsny.org>

Greetings Deer Taskforce.
I am, once again, soliciting your help for deer control in the areas behind Clinton Street Elementary, St Jude Terrace, Tim Tam Terrace and Organ Crescent. My home is on Organ Crescent.
The deer travel in packs ranging from 2 to 8 or 9. Recently on one occasion there were 12! They have made a complete joke of my few plants and sleeping all over my yard. I have a small tall double fenced garden where sunflowers and pumpkins are growing. While food is abundant for them in parks and other green spaces, they are leaning over my fencing, bending it terribly and aggressively trying to get inside. I am treating for deer ticks but they are in the yard multiple times nightly that I don't trust the efficacy of my deer tick prevention regime. I work often late into the night
and many nights find them obstructing traffic on Organ Crescent and Bosse Lane often barely moving out of the way of my car. They've stood in my driveway just looking at me, at times where I needed to get out of my car and lunge toward them just to get them out of the way. There were a couple of times this summer where they can right up on my deck next to my sliding glass door just to check the space out. There
are no plants on my deck for them. They did go up onto the deck of my next door neighbor and are all the lettuce he had growing. Deer on the deck of anyone's home is a mere reflection of their comfort being
all over everyone's property. It's too much!
This has worsened in abundance over the past couple of years. I am tired of them to no end. I am hoping you can help us with the terrible nuisance this has become. This neighborhood needs some help and support!
Please don't tell me all the reasons as to why they were all over in high numbers. Everyone knows all that. Everyone is practicing mitigating measures. The deer are obnoxious and there are entirely too many of them.
Help!
Thank you kindly,
Donna, Michael Jones
22 Organ Crescent

Walnut Off of Main - Deer Poop all over my lawn - People feeding deer!

Rick Deren <dric77@gmail.com>
Mon 8/23/2021 2:13 PM
To: Deer Task Force <WSDeerTaskForce@twsny.org>

Deer Task Force,

I moved into West Seneca last year from Lackawanna. We had a lot of deer there and never had problems. On my property I can at 5 or 5 deer an evening. They are nice and good for my 2 yr and 6 yr kids to see. My main issue is that they poop all over my lawn. I pickup pounds on a daily poop basis. The deer are not afraid and come up to you. People in my neighborhood feed the deer regularly, which I don't think is allowed. The door poor is a lot and usually liquidity. My main concern is the insects they carry, they lay all over my lawn.

Please direct me if there is anything. West Seneca has a program to help with this problem.

--
Thanks,

Rick Deren
38 Walnut Road
West Seneca, NY 14224
(716) 308-3869 (c)